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THE ASYMPTOTIC FORM OF THE SOLUTIONS OF THE INTEGRAL EQUATIONS OF POTENTIAL THEORY IN THE NEIGHBOURHOOD OF THE CORNER POINTS OF A CONTOUR*

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The integral equations of the theory of the logarithmic potential on a closed piecewise-smooth contour are considered. Asymptotic representations are obtained for the solutions of the integral equations in the region of the corner points of the contour, and formulas are obtained for the coefficients of these representations. As in /1/, information on the solutions of the integral equations of potential theory are derived from the well-known results of the asymptotic form of the solutions of the internal and external Dirichlet and Neumann problems. It is shown that, irrespective of the value of the angle in the region of the corner points of the contour, the solution of the integral equation of the internal Dirichlet problem has an unevenness, while the solution of the integral equation of the external Neumann problem has a singularity, whereas both the solutions of the boundary value Dirichlet and Neumann problems obtain irregularities in the region of the corner points only for angles occurring in the region.

Quite a number of asymptotic forms of the solutions of elliptic boundary value problems are known in the region of corner and conical points /2, 3/. However, the asymptotic behaviour of the solutions of the integral equations of the same problems have not been investigated, despite the need for such investigations for the method of boundary integral equations.

Below, using the example of the integral equations of the theory of the logarithmic potential, we describe a method for determining the asymptotic forms of the solutions in the region of the corner points of a contour. Only boundary value problems for harmonic functions of two variables are considered, but the scheme proposed is of a general form and can be applied to integral equations of other classical boundary value problems in mathematical physics.

Suppose Γ is a simple closed piecewise-smooth curve with a finite number of corner points $p_1, p_2, \ldots, p_m$. We will denote by Ω^i the boundary region, situated inside Γ, and by Ω^e the region external to Γ. We will denote the opening span of the angle between the semitangents at the points p_j from the side of the region Ω^i by α_j. We will assume that $0<\alpha_j<2\pi$ when $j=1, 2, \ldots, m$.

We will first consider the internal Dirichlet problem

$$\Delta u^i = 0 \quad \text{in } \Omega^i,\ u|_\Gamma = \varphi \tag{1}$$

where φ is the contraction of the smooth function in R^2 on the contour Γ. The classical method of solving (1) is to find the function in the form of the potential of a double layer

$$u(x) = -\int_\Gamma \mu(y)\frac{\cos\theta}{r}\,ds_y,\quad \mu\in C(\Gamma),\quad r=|x-y|$$

where θ is the angle formed by the external normal n_y to Γ at the point $y\neq p_j$ and the vector $y-x$. The density μ satisfies the integral equation

$$\mu(x)+\frac{1}{\pi}\int_\Gamma \mu(y)\frac{\cos\theta}{r}\,ds_y = -\frac{1}{\pi}\varphi(x),\quad x\in\Gamma\setminus Up_j \tag{2}$$

which is uniquely solvable in the space $C(\Gamma)$ /4/.

We will denote by u^e the solution of the external Dirichlet problem

$$\Delta u^e = 0 \quad \text{in } \Omega^e,\ u^e=\varphi \tag{3}$$

As is well-known

$$\varphi(x) = \frac{1}{2\pi}\int_\Gamma\left(\frac{\partial u^i}{\partial n}-\frac{\partial u^e}{\partial n}\right)\log\frac{1}{r}\,ds_y + u^e(\infty)\quad x\in\Gamma,\quad x\neq p_j \tag{4}$$

where $u^e(\infty)$ is the limiting value of the function u^e at infinity.

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We will introduce the solution v of the external Neumann problem

$$\Delta v = 0 \text{ in } \Omega^e, \ \frac{\partial v}{\partial n}\Big|_\Gamma = \frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n} \tag{5}$$

which tends to zero at infinity.

Since

$$v(x) = -\frac{1}{2\pi}\int_\Gamma \left(\log\frac{1}{r}\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}\log\frac{1}{r}\right)ds_y, \quad x \in \Omega^e$$

we have

$$v(x) + \frac{1}{\pi}\int_\Gamma v(y)\frac{\cos\theta}{r}ds_y = -\frac{1}{\pi}\int_\Gamma \left(\frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n}\right)\log\frac{1}{r}ds_y \quad x \in \Gamma\setminus Up_j$$

which, together with (4) shows that the function $(2\pi)^{-1}(v - u^e(\infty))$ is equal to the solution μ of (2).

For simplicity we will assume that close to the point p_j the region coincides with the sector $\{x_1 + ix_2 = \rho e^{i\omega};\ 0 < \rho < \delta,\ 0 < \omega < \alpha\}$.

Consider the case when $0 < \alpha < \pi$. As $\rho \to 0$ we have

$$u^i(x) = \varphi(0) + \frac{\partial\varphi}{\partial x_1}(o)x_1 + \frac{\partial\varphi}{\partial x_2}(0)x_2 + O(\rho^{1+\varepsilon}) \quad \varepsilon > 0, \quad x \in \Omega^i$$

$$u^e(x) = \varphi(0) + C_1\rho^\lambda\sin\lambda\omega + \frac{\partial\varphi}{\partial x_1}(0)x_1 + \frac{\partial\varphi}{\partial x_2}(0)x_2 + O(\rho^{1+\varepsilon}), \qquad \lambda = \frac{\pi}{2\pi - \alpha}, \quad x \in \Omega^e$$

Since these equations can be differentiated, we have

$$\frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n} = \lambda C_1\rho^{\lambda-1} + O(\rho^\varepsilon), \quad x \in \Gamma\setminus p_j, \quad \rho \to 0$$

Hence it follows from (5) that

$$v(x) - v(0) \sim C_0\rho^\lambda\cos\lambda\omega - C_1\rho^\lambda\sin\lambda\omega, \ x \in \Omega^e, \ \rho \to 0 \tag{6}$$

Hence, when $0 < \alpha < \pi$ we have

$$\mu(x) - \mu(0) \sim \pm\frac{1}{2\pi}C_0\rho^\lambda, \quad \rho \to 0 \tag{7}$$

where the plus sign corresponds to the ray $\omega = 0$, and the minus sign corresponds to the ray $\omega = 2\pi - \alpha$.

The constant C_0 is determined by the method described in /2/. We will denote by η the function from $C^\infty(0,\infty)$, $\eta(\rho) = 1$ when $\rho < \delta$, $\eta(\rho) = 0$ when $\rho > 2\delta$. We will put

$$w(x) = v(x) - v(0) + C_1\eta(\rho)\rho^\lambda\sin\lambda\omega \tag{8}$$

According to (5), this function is the solution of the problem

$$\Delta w(x) = -C_1\Delta(\eta(\rho)\rho^\lambda\sin\lambda\omega), \ x \in \Omega^e$$

$$\frac{\partial w}{\partial n}\Big|_\Gamma = \frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n} + C_1\frac{\partial}{\partial n}(\eta(\rho)\rho^\lambda\sin\lambda\omega)$$

In view of (6) and (8) we have

$$w(x) \sim C_0\rho^\lambda\cos\lambda\omega, \ x \in \Omega^e, \ \rho \to 0$$

Hence (see /2/) the following equation holds:

$$C_0 = -\frac{1}{\pi}\int_{\Omega^e}\zeta^e C_1\Delta(\eta(\rho)\rho^\lambda\sin\lambda\omega)\,dx + \frac{1}{\pi}\int_\Gamma \zeta^e\left\{\frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n} + C_1\frac{\partial}{\partial n}(\eta(\rho)\rho^\lambda\sin\lambda\omega)\right\}ds$$

where ζ^e is an harmonic function in Ω, continuous outside any neighbourhood of the point p_j with zero normal derivative on $\Gamma\setminus Up_j$, and having the asymptotic form

$$\zeta^e(x) \sim \rho^{-\lambda}\cos\lambda\omega, \ \rho \to 0$$

Hence, after simplification we obtain

$$C_0 = \frac{1}{\pi}\int_\Gamma \zeta^e\frac{\partial u^i}{\partial n}ds \tag{9}$$

We will construct an harmonic extension Z^i of the function ζ^e in the region Ω^i while preserving the continuity on $\Gamma\setminus p_j$. Then, in view of (9) we have

$$C_0 = \frac{1}{\pi} \int_\Gamma [\varphi - \varphi(0)] \frac{\partial Z^i}{\partial n} ds \tag{10}$$

Equations (7) and (10) define the principal term of the asymptotic form of the function $\mu - \mu(0)$ when $0 < \alpha < \pi$.

Suppose now that $\pi < \alpha < 2\pi$. When $x \in \Omega^i$ and $\delta \to 0$, the solution of (1) and (3) have the asymptotic form

$$u^i(x) = \varphi(0) + D_1\rho^\sigma \sin\sigma\omega + \frac{\partial\varphi}{\partial x_1}(0)\, x_1 + \frac{\partial\varphi}{\partial x_2}(0)\, x_2 + O(\rho^{1+\varepsilon}), \quad \sigma = \frac{\pi}{\alpha}$$

$$u^e(x) = \varphi(0) + \frac{\partial\varphi}{\partial x_1}(0)\, x_1 + \frac{\partial\varphi}{\partial x_2}(0)\, x_2 + O(\rho^{1+\varepsilon})$$

Hence

$$\frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n} = -\sigma D_1 \rho^{\sigma-1} + O(\rho^\varepsilon)$$

Consequently,

$$v(x) - v(0) \sim D_1\rho^\sigma \frac{\sin\sigma(\omega-\pi)}{\cos\sigma\pi}, \quad \alpha \leqslant \omega \leqslant 2\pi, \quad x \in \Omega^e, \quad \rho \to 0$$

Hence, bearing in mind the relationship between the functions v and μ, we find that

$$\mu(x) - \mu(0) \sim \frac{\operatorname{tg}\sigma\pi}{2\pi} D_1\rho^\sigma$$

The constant D_1 is defined as follows /2/:

$$D_1 = -\frac{1}{\pi} \int_\Gamma [\varphi(x) - \varphi(0)] \frac{\partial \zeta^i}{\partial n} ds \tag{11}$$

where ζ^i is a function harmonic in Ω^i, equal to zero on $\Gamma \setminus p_j$ and having the asymptotic form $\zeta^i \sim \rho^{-\sigma} \sin\sigma\omega$.

We will now consider the asymptotic form of the solution, conjugate to (2), of the integral equation of the external Neumann problem

$$\Delta v^e = 0 \text{ in } \Omega^e, \frac{\partial v^e}{\partial n} = \psi \text{ on } \Gamma \setminus U p_j \tag{12}$$

where ψ is a function that is smooth on the closed arcs $\widehat{p_1p_2}, \ldots, \widehat{p_{m-1}p_m}, \widehat{p_mp_1}$. If

$$v^e(x) = \int_\Gamma \nu(y) \log\frac{1}{r} ds_y$$

we have

$$\nu(x) - \frac{1}{\pi}\int_\Gamma \nu(y) \frac{\partial}{\partial n_x} \log\frac{1}{r} ds_y = -\frac{1}{\pi}\psi(x), \quad x \in \Gamma \setminus U p_j$$

The asymptotic form of the density in the region of the corner point p_j can be found from the equation

$$\nu(x) = \frac{1}{2\pi}\left(\frac{\partial v^i}{\partial n} - \frac{\partial v^e}{\partial n}\right) \tag{13}$$

where v^i is the solution of the Dirichlet problem

$$\Delta v^i = 0 \text{ in } \Omega^i, \ v^i = v^e \text{ on } \Gamma \tag{14}$$

Suppose $0 < \alpha < \pi$. The solutions of (12) and (14) have the asymptotic form

$$v^e(x) - v^e(0) \sim C_2\rho^\lambda \cos\lambda(\omega - \alpha), \ \alpha \leqslant \omega \leqslant 2\pi$$

$$v^i(x) - v^e(0) \sim C_2\rho^\lambda \sin\lambda\left(\omega - \frac{\alpha}{2}\right)\left(\sin\frac{\lambda\alpha}{2}\right)^{-1}, \quad 0 \leqslant \omega \leqslant \alpha$$

Hence, using (13) we obtain

$$\nu \sim \pm[2(2\pi-\alpha)]^{-1} C_2\rho^{\lambda-1} \operatorname{ctg}\frac{\lambda\alpha}{2}$$

where the plus sign corresponds to the ray $\omega = \alpha$, and the minus sign corresponds to the ray $\omega = 0$. The constant C_2 is given by /2/

$$C_2 = \frac{1}{\pi}\int_\Gamma \zeta^e \psi\, ds$$

where ζ^e is the same function as in (9).

Consider the case when $\pi < \alpha < 2\pi$. The solutions of (12) and (14) have the asymptotic form

$$v^e(x) - v^e(0) = O(\rho) \qquad (15)$$
$$v^i(x) - v^e(0) \sim C_1\rho^{\sigma}\sin\sigma\omega,\ 0 \leqslant \omega \leqslant \alpha$$

Hence, from (13) we obtain

$$\nu \sim (2\alpha)^{-1} C_1\rho^{\sigma-1},\quad C_1 = -\frac{1}{\pi}\int\limits_{\Gamma}[v^e - v^e(0)]\frac{\partial\zeta^i}{\partial n}\,ds \qquad (16)$$

where ζ^i is the same function as in (11).

We will introduce the solution Z^e of the Neumann problem

$$\Delta Z^e = 0 \ \text{ in } \ \Omega^e,\ \frac{\partial Z^e}{\partial n} = \frac{\partial\zeta^i}{\partial n} \quad \text{on} \quad \Gamma\setminus Up_j \qquad (17)$$

We will show that a solution Z^e of problem (17) exists, which vanishes at infinity, and in the region of the point p_j has the asymptotic form

$$Z^e(x) = \rho^{-\sigma}\frac{\sin\sigma(\omega-\pi)}{\cos\sigma\pi} + O(1) \qquad (18)$$

For small ρ we have

$$\zeta^i(x) = \rho^{-\sigma}\sin\sigma\omega + O(\rho^{\sigma}),\ 0 \leqslant \omega \leqslant \alpha \qquad (19)$$

We will denote by g and h the first terms on the right sides of (18) and (19) respectively. It is obvious that $\partial g/\partial n = \partial h/\partial n$ on Γ in the region of the point p_j. We will put $Z^e = \eta g - \Theta$, where η is the function introduced earlier, and Θ is the solution of the Neumann problem

$$\triangle\Theta = 2\nabla\eta\nabla g + g\triangle\eta \ \text{ in } \Omega^e,\quad \frac{\partial\Theta}{\partial n} = \frac{\partial}{\partial n}(\eta g - \zeta^i)$$

on $\Gamma\setminus p_j$ with finite Dirichlet integral.

The function Z^e, constructed in this way, is unique, apart from a constant term and, possibly, has a logarithmic increase at infinity. We will show that this increase does not in fact occur.

We will denote by $\gamma_\varepsilon{}^i$ and $\gamma_\varepsilon{}^e$ the parts of the circle γ_ε of radius ε with centre at the point p_j situated in Ω^i and Ω^e respectively, where ε is a small positive number. Suppose also that $\Gamma_\varepsilon = \Gamma\setminus K_\varepsilon$, where K_ε is a circle of radius ε with centre at the point p_j and γ_R is a circle of fairly large radius R, inside which Γ is situated. It is obvious that

$$\int\limits_{\gamma_R}\frac{\partial Z^e}{\partial\rho}\,ds = -\int\limits_{\gamma_\varepsilon{}^e}\frac{\partial Z^e}{\partial\rho}\,ds + \int\limits_{\Gamma_\varepsilon}\frac{\partial Z^e}{\partial n}\,ds = -\int\limits_{\gamma_\varepsilon{}^e}\frac{\partial g}{\partial\rho}\,ds + \int\limits_{\Gamma_\varepsilon}\frac{\partial\zeta^i}{\partial n}\,ds + O(\varepsilon^{\sigma})$$

We can show by a direct check that

$$\int\limits_{\gamma_\varepsilon{}^e}\frac{\partial g}{\partial\rho}\,ds = \int\limits_{\gamma_\varepsilon{}^i}\frac{\partial h}{\partial\rho}\,ds = -2\varepsilon^{-\sigma}$$

Hence, the flow of the solution Z^e through γ_R is .

$$\int\limits_{\gamma_R}\frac{\partial Z^e}{\partial\rho}\,ds = \int\limits_{\gamma_\varepsilon{}^i}\frac{\partial h}{\partial n}\,ds + \int\limits_{\Gamma_\varepsilon}\frac{d\zeta^i}{\partial n}\,ds + O(\varepsilon^{\sigma}) = \int\limits_{\Gamma_\varepsilon\cup\gamma_\varepsilon{}^i}\frac{\partial\zeta^i}{\partial n}\,ds + O(\varepsilon^{\sigma}) = O(\varepsilon^{\sigma})$$

and at infinity the function Z^e approaches a constant, which can be put equal to zero.

We will now express the constant in (16) in terms of the boundary values of the solution of problem (12). In view of (15)—(17) we have

$$-\pi C_1 = \int\limits_{\Gamma_\varepsilon\cup\gamma_\varepsilon{}^e}[v^e - v^e(0)]\frac{\partial Z^e}{\partial n}\,ds + O(\varepsilon^{1-\sigma})$$

Using Green's formula we obtain

$$-\pi C_1 = -\int\limits_{\gamma_R}[v^e - v^e(0)]\frac{\partial Z^e}{\partial n}\,ds + \int\limits_{\gamma_R} Z^e\frac{\partial v^e}{\partial n}\,ds + \int\limits_{\Gamma_\varepsilon} Z^e\frac{\partial v^e}{\partial n}\,ds - \int\limits_{\gamma_\varepsilon{}^e} Z^e\frac{\partial v^e}{\partial\rho}\,ds + O(\varepsilon^{1-\sigma})$$

Passing to the limit in this expression as $R \to \infty$ and $\varepsilon \to 0$, and using the first relation from (15) and (18), we see that the integrals over γ_R and γ_ε^e tend to zero. Hence

$$C_1 = -\frac{1}{\pi}\int Z^e \psi \, ds$$

and consequently the solution ν of the integral equation of the external Neumann problem has the asymptotic form

$$\nu(x) \sim -(2\alpha\pi)^{-1}\rho^{\sigma-1}\int_\Gamma Z^e \psi \, ds$$

The asymptotic form of the solutions of the integral equations of the external Dirichlet problem and the internal Neumann problem can be found similarly.

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